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SPECIFICATION TEST

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# A CONSISTENT HAUSMAN-TYPE MODEL SPECIFICATION TEST

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*Abstract*: In this paper we propose a modification of White's (1981) version of Hausman's (1978) model specification test such that the test has asymptotic power 1 against *any* deviation from the null hypothesis that the model is correctly specified.

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## 1. INTRODUCTION

In an influential paper in *Econometrica*, Hausman (1978) proposed a test for model misspecification. This test is based on the idea that for a true model the difference between an efficient estimator and an inefficient estimator, times the square root of the sample size, will converge in distribution to the normal with zero mean, whereas under the alternative that the model is misspecified it is likely that the two estimators will deviate in the limit. White's (1981) version of this test compares the ordinary nonlinear least squares estimator with the weighted nonlinear least squares estimator. See also Ruud (1984) for a review of Hausman-type model specification tests.

The power of the Hausman test depends heavily on the choice of the inefficient estimator or, in White's case, on the weight function. Bierens (1982, p.128) illustrates the latter by the following example. Let the true model be

$$y_j = x_{1j} + x_{2j} + x_{1j}x_{2j} + u_j ,$$

where the $x_{ij}$'s and the $u_j$'s are independently N(0,1) distributed. If this model is specified as

$$y_j = \theta_1 x_{1j} + \theta_2 x_{2j} + u_j$$

and if White's version of Hausman's test is conducted with weight function

$$w(x_{1j}, x_{2j}) = x_{1j}^2 + x_{2j}^2 ,$$

then the test statistic involved converges in distribution to $(13/2)\chi^2(2)$. The asymptotic power of the test is therefore less than 1. Although this example is rather artificial and does not bear much relevance to econometrics, it shows at least that the test is not watertight.

The problem of lack of consistency is not typical for Hausman's test, but is common to virtually all model specification tests. Tests which are designed to test a null against a specific alternative may have low power against other alternatives. Although Hausman's test is not designed to test against a specific alternative there exists in fact an implicit alternative, depending on the choice of the inefficient estimator, against which it has maximal power. Again the test may have low power against deviations from this implicit alternative. See Holly (1982).

To the best of our knowledge the only consistent model specification tests are those of Bierens (1982, 1984, 1986a,b). These tests test the null hypothesis that the response function of the model equals the conditional expectation of the dependent variable relative to the regressors (and all lagged dependent variables and lagged regressors in the time series case). The alternative against which these tests have asymptotic power 1 is the general alternative that the null is false. Thus, *any* deviation from the null will be (asymptotically) detected. The tests of Bierens (1982, 1984), however, have some serious drawbacks. First, the distribution of the test statistic under the null is of an unknown form, so that the critical level had to be determined by using Chebishev's inequality. Second, these tests are quite laborious. The tests of Bierens (1986a,b) are consistent standard normal tests, but they rely on the assumption that the data generating process is rational-valued or is truncated rational-valued. Although in practice economic data are always rational-valued, we shall not employ this property here.

In the present paper it will be shown that White's version of Hausman's test can be converted into a consistent test, i.e., we shall propose a class of weight functions such that the test has asymptotic power 1 against all deviations from the null.

In Section 2 we outline the test and the ideas behind it. In Section 3 we set forth (mild) maintained hypotheses for the consistency of the test. Basic to our approach is a generalisation of Theorems 1 and 2 of Bierens (1982), which is stated as a Lemma in Section 3 and proved in the Appendix. In Section 4 we briefly discuss the asymptotic distribution of the test statistic.

Throughout the paper we assume an i.i.d. data generating process. Using the approach in Bierens (1981, Ch.3, 1984, 1986a,b), however, it seems feasable to extend the results to the heterogeneous and/or time series case. These extensions are left for future research.

## 2. THE TEST

Let $\{(y_1,x_1),\ldots,(y_n,x_n)\}$ be a sample from a k+1-variate distribution, where $y_j \in R$ (and thus $x_j \in R^k$). We consider a nonlinear regression model:

$$y_j = f(x_j,\theta_0) + u_j\,,\ \theta_0 \in \Theta,$$

where $\Theta$ is a *compact* and *convex* subset of $R^m$ and $f(x,\theta)$ is for each $\theta \in \Theta$ a Borel measurable real function on $R^k$ and for each k-vector x a *continuously differentiable* real function on $\Theta$.

In order that this model is correctly specified the errors $u_j$ should satisfy

$$E(u_j|x_j) = 0 \text{ a.s.}$$

This is the null hypothesis to be tested, or equivalently

(1) $H_0$: $P[E(y_j|x_j) = f(x_j,\theta_0)] = 1$ for some $\theta_0 \in \Theta$.

The alternative hypothesis is that the null is false, i.e.,

(2) $H_1$: $P[E(y_j|x_j) = f(x_j,\theta)] < 1$ for all $\theta \in \Theta$.

In this paper we propose a consistent Hausman-type test of $H_0$ against $H_1$, by comparing the nonlinear least squares estimator $\hat{\theta}$ for $\theta_0$ with a weighted nonlinear least squares estimator $\hat{\theta}(t)$, where $t \in R^k$ is a vector of nuisance parameters of the weight function. Thus we consider White's version of Hausman's test, but with a weight function guaranteeing consistency. Moreover our null is slightly more general in that $H_0$ does not imply that the $u_j$'s are independent of $x_j$ or that $E(u_j^2|x_j) = E(u_j^2)$ a.s.

The estimator $\hat{\theta}(t)$ is obtained by minimizing

(3) $$\hat{Q}(\theta,t) = (1/n)\textstyle\sum_{j=1}^{n}[y_j - f(x_j,\theta)]^2\exp(t'\Phi(x_j))$$

over the parameter space $\Theta$, where $\Phi$ is a *bounded Borel measurable one-to-one mapping* from $R^k$ into $R^k$. For example, we may choose

$$\Phi(z) = \Phi(z_1,..,z_k) = (\text{atan}(z_1),..,\text{atan}(z_k))'; \qquad z = (z_1,..,z_k)' \in R^k.$$

Thus $\hat{\theta}(t)$ is for given non-random $t \in R^k$ a measurable solution of

$$\hat{\theta}(t) \in \Theta: \hat{Q}(\hat{\theta}(t),t) = \inf_{\theta \in \Theta} \hat{Q}(\theta,t).$$

Obviously, the ordinary nonlinear least squares estimator $\hat{\theta}$ equals $\hat{\theta}(0)$. Now let $\theta(t)$ be the probability limit of $\hat{\theta}(t)$:

$$\theta(t) = \text{plim}_{n\to\infty}\hat{\theta}(t),$$

and let S be the following subset of $R^k$:

(4) $S = \{ t \in R^k: \theta(t) = \theta(0)\}$.

We shall set forth mild conditions such that under $H_1$ this set S has *Lebesgue measure zero*, or more generally that S is a *null set* with respect to any probability measure $\mu$ on $R^k$ induced by an absolutely continuous k-variate distribution:

$$\mu(S) = 0.$$

On the other hand, if $H_0$ is true then clearly

$$\mu(S) = 1,$$

and moreover we then have

$$\sqrt{n}\ (\hat{\theta}(t) - \hat{\theta}(0)) \to N_m[0,\Omega(t)] \text{ in distr.}$$

for every $t \in R^k$. Furthermore, we shall propose an estimator $\hat{\Omega}(t)$ such that

$$\text{plim}_{n\to\infty}\hat{\Omega}(t) = \Omega(t)$$

under $H_0$ as well as under $H_1$, although in the latter case $\Omega(t)$ is no longer the variance matrix of the asymptotic distribution of $\hat{\theta}(t)$. Now define

$$\hat{W}(t) = n(\hat{\theta}(t)-\hat{\theta}(0))'\hat{\Omega}(t)^{-1}(\hat{\theta}(t)-\hat{\theta}(0))$$

and assume that $\Omega(t)$ is nonsingular for $t \neq 0$. Then

$$\hat{W}(t) \to \chi_m^{\ 2} \text{ in distr. if } H_0 \text{ is true and } t \neq 0,$$

whereas

$$\text{plim}_{n\to\infty}\hat{W}(t)/n = (\theta(t)-\theta(0))'\Omega(t)^{-1}(\theta(t)-\theta(0)) > 0$$
$$\text{if } H_1 \text{ is true and } t \notin S.$$

The latter result implies, of course, that under $H_1$,

$$\text{plim}_{n\to\infty}\hat{W}(t) = \infty \text{ if } t \notin S.$$

This result establishes the 'almost' consistency of the test under review, i.e., consistency fails only for t in a set S with Lebesgue measure zero.

In practice the set S is unknown, and although S is 'almost' empty we will never be sure whether a given fixed t is in S or not (except $t = 0$). But even if we would be sure that $t \notin S$ so that the asymptotic power of the test equals 1, the small sample power of the test may vary substantially with t. Thus, we might think of maximizing $\hat{W}(t)$ over some subset T of $R^k$, and using the resulting $\hat{t}$ instead of t. However, $\hat{W}(\hat{t})$ will then no longer be asymptotically $\chi_m^2$ distributed under $H_0$. So optimizing the small sample power of the test seems not feasable. The only way to guarantee that t lies outside S is to draw t randomly from an absolutely continuous k-variate distribution, for example the k-variate standard normal distribution, as then $P[t \in S] = 0$. Denoting such a random drawing by $\hat{t}$ we then have, similarly to the results in Bierens (1986a,b),

$$\hat{W}(\hat{t}) \to \chi^2_m \quad \text{in distr. under } H_0$$

and

$$\text{plim}_{n\to\infty}\hat{W}(\hat{t}) = \infty \text{ under } H_1 .$$

Thus by replacing the fixed t with this random $\hat{t}$ we get a $\chi^2$ test which has asymptotic power 1 against *any* deviation from the null hypothesis.

## 3. CONSISTENCY OF THE TEST

Throughout this section we assume that the alternative hypothesis (3) is true and thus that the model is misspecified. Moreover, we shall state a number of other assumptions which are to hold under the null as well. These assumptions are parts of the maintained hypothesis and are quite mild. The first one guarantees that $\hat{Q}(\theta,t)$ converges uniformly on $\Theta$ to its mathematical expectation.

ASSUMPTION 1. *Let* $E\, y_1^2 < \infty$ *and* $E \sup_{\theta\in\Theta} f(x_1,\theta)^2 < \infty$ .

Then it follows from Theorem 2 of Jennrich (1969) that for each $t \in R^k$,

$$\sup_{\theta\in\Theta}|\hat{Q}(\theta,t) - Q(\theta,t)| \to 0 \text{ a.s.},$$

where

$$Q(\theta,t) = E\,\hat{Q}(\theta,t) = E\,[y_1-f(x_1,\theta)]^2\exp(t'\Phi(x_1)).$$

Clearly this limit function $Q(\theta,t)$ is continuous in both arguments. Since $\Theta$ is compact there exists for each t a $\theta(t)$ such that

$$\theta(t) \in \Theta : Q(\theta(t),t) = \inf_{\theta \in \Theta} Q(\theta,t).$$

Now we assume:

ASSUMPTION 2. *For each* $t \in R^k$, $\theta(t)$ *is unique.*

Then it follows from Lemma 3.1.8. of Bierens (1981) that for each $t \in R^k$

$$\hat{\theta}(t) \to \theta(t) \text{ a.s.}$$

Moreover, it can even be shown that the a.s. convergence involved is uniform on any compact subset of $R^k$.

Next, assume:

ASSUMPTION 3. *For each* $t \in R^k$ *let* $\theta(t)$ *be an interior point of* $\Theta$.

Then the first order condition for a minimum of $Q(\theta,t)$ applies, i.e.

$$(\partial/\partial\theta')Q(\theta,t)\Big|_{\theta=\theta(t)} = 0,$$

hence

$$(5) \quad E[y_1-f(x_1,\theta(t))](\partial/\partial\theta')f(x_1,\theta(t))\exp(t'\Phi(x_1)) = 0.$$

Now let $S^*$ be the set

$$(6) \quad S^* = \{t \in R^k: E[y_1-f(x_1,\theta(0))](\partial/\partial\theta')f(x_1,\theta(0))\exp(t'\Phi(x_1)) = 0\ \}.$$

Suppose that for some $t \notin S^*$, $\theta(t) = \theta(0)$. Then (5) contradicts with $t \notin S^*$, hence $S \subset S^*$, where $S$ is defined in (4). Thus for proving $\mu(S) = 0$ it suffices to show $\mu(S^*) = 0$, and for that we need the following fundamental lemma:

LEMMA. *Let v be a random variable satisfying* $E|v| < \infty$ *and let* x *be a bounded random vector in* $R^k$ *such that*

$$P[E(v|x) = 0] < 1.$$

*Consider the set*

$$S = \{ t \in R^k : Ev.\exp(t'x) = 0 \}.$$

*For every probability measure* $\mu$ *induced by an absolutely continuous distribution on* $R^k$ *we have:*

$$\mu(S) = 0.$$

*Proof*: Appendix.

Since the model is misspecified, we have

$$P\{E[y_1 - f(x_1,\theta(0))|x_1] = 0\} < 1,$$

hence,

$$P\{E[(y_1 - f(x_1,\theta(0))(\partial/\partial\theta')f(x_1,\theta(0))|x_1] = 0\} < 1.$$

Moreover, since $\Phi$ is a one-to-one mapping conditioning on $x_1$ is equivalent to conditioning on the bounded random vector $\Phi(x_1)$, hence

$$\text{(7)} \quad P\{E[(y_1 - f(x_1,\theta(0))(\partial/\partial\theta')f(x_1,\theta(0))|\Phi(x_1)] = 0\} < 1.$$

It follows now from (7) and the Lemma that $\mu(S^*)=0$ and thus:

THEOREM 1. *Under Assumptions 1-3 and* $H_1$ *the set* $S = \{t \in R: \theta(t) \neq \theta(0)\}$ *is a null set with respect to any probability measure* $\mu$ *induced by an absolutely continuous k-variate distribution.*

## 4. FURTHER ASYMPTOTIC THEORY

The asymptotic theory of the test statistic $\hat{W}(t)$ under $H_0$ is quite standard (see White 1981) and will therefore be discusses only briefly. Throughout we assume that Assumptions 1-3 hold. Then under $H_0$

$$\theta(t) = \theta(0) = \theta_0 \text{ for all } t \in R^k$$

and consequently,

$$\hat{\theta}(t) \to \theta_0 \text{ a.s., for all } t \in R^k .$$

Now denote

$$\hat{\Omega}_1(\theta,t) = (1/n)\textstyle\sum_{j=1}^n \{(\partial/\partial\theta')f(x_j,\theta)\}\{(\partial/\partial\theta)f(x_j,\theta)\}\exp(t'\Phi(x_j)),$$

$$\hat{\Omega}_2(\theta,t) = (1/n)\textstyle\sum_{j=1}^n (y_j-f(x_j,\theta))^2\{(\partial/\partial\theta')f(x_j,\theta)\}\{(\partial/\partial\theta)f(x_j,\theta)\} \times \exp(t'\Phi(x_j)),$$

$$\hat{\Gamma}(\theta,t) = (1/n)\textstyle\sum_{j=1}^n (y_j- f(x_j,\theta))(\partial/\partial\theta)(\partial/\partial\theta')f(x_j,\theta)\exp(t'\Phi(x_j))$$

and let the following assumption hold.

ASSUMPTION 4. *For $i_1, i_2 = 1,..,m$*

$$E \sup_{\theta\in\Theta}|\{(\partial/\partial\theta_{i_1})f(x_1,\theta)\}\{(\partial/\partial\theta_{i_2})f(x_1,\theta)\}| < \infty,$$

$$E \sup_{\theta\in\Theta}|(y_1-f(x_1,\theta))^2\{(\partial/\partial\theta_{i_1})f(x_1,\theta)\}\{(\partial/\partial\theta_{i_2})f(x_1,\theta)\}| < \infty,$$

$$E \sup_{\theta\in\Theta}|(y_1-f(x_1,\theta))(\partial/\partial\theta_{i_1})(\partial/\partial\theta_{i_2})f(x_1,\theta)| < \infty.$$

Then it follows from Jennrich (1969, Theorem 2) that for each $t \in R^k$,

$$\hat{\Omega}_1(\theta,t) \to E\,\hat{\Omega}_1(\theta,t) \text{ a.s., uniformly on } \Theta,$$

$$\hat{\Omega}_2(\theta,t) \to E\,\hat{\Omega}_2(\theta,t) \text{ a.s., uniformly on } \Theta,$$

$$\hat{\Gamma}(\theta,t) \to E\,\hat{\Gamma}(\theta,t) \text{ a.s., uniformly on } \Theta.$$

Thus:

$$\begin{aligned}(\partial/\partial\theta)(\partial/\partial\theta')\hat{Q}(\theta,t) &= 2\hat{\Omega}_1(\theta,t) - 2\hat{\Gamma}(\theta,t) \\ &\to E\,(\partial/\partial\theta)(\partial/\partial\theta')\hat{Q}(\theta,t) = 2\,E\,\hat{\Omega}_1(\theta,t) - 2\,E\,\hat{\Gamma}(\theta,t) \text{ a.s.}\end{aligned}$$

uniformly on $\Theta$. Moreover, by the central limit theorem we have

$$(8)\quad \sqrt{n}\begin{pmatrix}(\partial/\partial\theta')\hat{Q}(\theta_0,t)\\ (\partial/\partial\theta')\hat{Q}(\theta_0,0)\end{pmatrix} \to N_{2m}\left(0\ ,\ \begin{pmatrix}E\hat{\Omega}_2(\theta_0,2t) & E\hat{\Omega}_2(\theta_0,t)\\ E\hat{\Omega}_2(\theta_0,t) & E\hat{\Omega}_2(\theta_0,0)\end{pmatrix}\right)$$

Consequently, if

ASSUMPTION 5. $E\hat{\Omega}_1(\theta_0,t)$ *is nonsingular, for all* $t \in R^k$,

then by the standard Taylor expansion approach and the fact that

$$E\,\hat{\Gamma}(\theta_0,t) = 0$$

it follows

$$\sqrt{n}\begin{pmatrix}\hat{\theta}(t)-\theta_0\\ \hat{\theta}(0)-\theta_0\end{pmatrix} \to N_{2m}\left(0,\begin{pmatrix}\Omega_1(t)^{-1}\Omega_2(2t)\Omega_1(t)^{-1}, \Omega_1(t)^{-1}\Omega_2(t)\Omega_1(0)^{-1}\\ \Omega_1(0)^{-1}\Omega_2(\ t)\Omega_1(t)^{-1}, \Omega_1(0)^{-1}\Omega_2(0)\Omega_1(0)^{-1}\end{pmatrix}\right),$$

where

$$(9) \quad \Omega_1(t) = E\hat{\Omega}_1(\theta(0),t) \quad ,\Omega_2(t) = E\hat{\Omega}_2(\theta(0),t)$$

It is now easy to show:

THEOREM 2. *Under Assumptions 1-5 and* $H_0$ *we have*

$$\sqrt{n}(\hat{\theta}(t) - \hat{\theta}(0)) \to N_m[0,\Omega(t)] \text{ in distr.}$$

*where*

$$\Omega(t) = \Omega_1(t)^{-1}\Omega_2(2t)\Omega_1(t)^{-1} - \Omega_1(t)^{-1}\Omega_2(t)\Omega_1(0)^{-1}$$
$$- \Omega_1(0)^{-1}\Omega_2(t)\Omega_1(t)^{-1} + \Omega_1(0)^{-1}\Omega_2(0)\Omega_1(0)^{-1}$$

*with* $\Omega_1(t)$ *and* $\Omega_2(t)$ *defined in (9).*

Moreover, denoting

$$\hat{\Omega}(t) = \hat{\Omega}_1(\hat{\theta}(0),t)^{-1}\hat{\Omega}_2(\hat{\theta}(0),2t)\hat{\Omega}_1(\hat{\theta}(0),t)^{-1}$$
$$- \hat{\Omega}_1(\hat{\theta}(0),t)^{-1}\hat{\Omega}_2(\hat{\theta}(0),t)\hat{\Omega}_1(\hat{\theta}(0),0)^{-1}$$
$$- \hat{\Omega}_1(\hat{\theta}(0),0)^{-1}\hat{\Omega}_2(\hat{\theta}(0),t)\hat{\Omega}_1(\hat{\theta}(0),t)^{-1}$$
$$+ \hat{\Omega}_1(\hat{\theta}(0),0)^{-1}\hat{\Omega}_2(\hat{\theta}(0),0)\hat{\Omega}_1(\hat{\theta}(0),0)^{-1}$$

we have:

THEOREM 3. *Under Assumptions 1-5,* $\hat{\Omega}(t) \to \Omega(t)$ *a.s., for each* $t \in R^k$.

Note that this result holds under $H_0$ as well as under $H_1$.

Next, assume:

ASSUMPTION 6. *$\Omega(t)$ is nonsingular for* $t \neq 0$.

A sufficient condition for this is that the asymptotic variance matrix in (8) is nonsingular for each $t \neq 0$ and that Assumption 5 holds. This assumption is of a high level and therefore not quite satisfactory. On the other hand it is not implausible an assumption and moreover it is basically the same assumption as the one in White (1981, Theorem 4.1).

It follows now from Theorems 1, 2 and 3:

THEOREM 4. *Under Assumptions 1-6 we have*

(10) $\hat{W}(t) \to \chi^2_m$ *in distr. if* $H_0$ *is true and* $t \neq 0$,

*whereas*

(11) $\text{plim}_{n\to\infty}\hat{W}(t)/n = (\theta(t)-\theta(0))'\Omega(t)^{-1}(\theta(t)-\theta(0)) > 0$

*if* $H_1$ *is true and* $t \notin S$.

Finally, let $\hat{t}$ be a random drawing from an absolutely continuous k-variate distribution with density $h(t)$. Then under $H_0$ the characteristic function of $\hat{W}(\hat{t})$ will converge to the characteristic function of $\chi^2_m$:

$$E \exp(i\zeta\hat{W}(\hat{t})) = E\{E[\exp(i\zeta\hat{W}(\hat{t}))|\hat{t}]\} = \int E \exp(i\zeta\hat{W}(t))h(t)dt$$
$$\to E \exp(i\zeta\chi^2_m)$$

by bounded convergence and (10). Moreover, denoting the right hand side of (11) by $\eta(t)$ we have under $H_1$,

$$\hat{W}(\hat{t})/n \to \eta(\hat{t}) \text{ in distr. and } P[\eta(\hat{t})>0] = P[\hat{t}\notin S] = 1,$$

hence

$$\text{plim}_{n\to\infty}\hat{W}(\hat{t}) = \infty$$

Thus:

THEOREM 5. *Let $\hat{t}$ be a random drawing from an absolutely continuous k-variate distribution. Moreover, let Assumptions 1-6 hold. Then*

$$\hat{W}(\hat{t}) \to \chi_m^2 \textit{ in distr. under } H_0,$$

and

$$\hat{W}(\hat{t}) \to \infty \textit{ in prob. under } H_1.$$

**APPENDIX: Proof of the Lemma**

First, let k=1. According to Theorem 2 of Bierens (1982) there exists a nonnegative integer m such that $Ev.x^m \neq 0$, hence

$$(d/dt)^m Ev.e^{tx} = \sum_{j=0}^{\infty}(d/dt)^m Ev(tx)^j/j! = \sum_{j=m}^{\infty} t^{j-m} Ev.x^j/(j-m)!$$
$$\rightarrow Ev.x^m \neq 0 \text{ as } t \rightarrow 0.$$

This implies that $Ev.\exp(tx) \neq 0$ in a neighborhood of zero. Now let $t_0$ be such that $Ev.\exp(t_0x) = 0$. Since

$$P(E[v.\exp(t_0x)|x] = 0\ ) = P(E(v|x) = 0\ ) < 1$$

it follows from the above argument, with v replaced by $v.\exp(t_0x)$, that $Ev.\exp(t_0x+tx) \neq 0$ in a neighborhood of $t = 0$, hence $Ev.\exp(tx) \neq 0$ in a neighborhood of $t = t_0$. This implies

(A1) $\inf_{t\in S,\, t\neq t_0}|t-t_0| > 0$ if $t_0 \in S$

and in its turn (A1) implies that S is *countable*. Since a countable set has zero probability with respect to any continuous distribution, the lemma follows for the case $k = 1$.

Next, consider the general case $k \geq 1$. Let

$$x = (x_1,x_2,..,x_k)';\ t = (t_1,t_2,..,t_k)';\ \hat{t} = (\hat{t}_1,\hat{t}_2,..,\hat{t}_k)',$$

where $\hat{t}$ is a random drawing from $\mu$. Again it follows from Theorem 2 of Bierens (1982) that there exist nonnegative integers $m_1,..,m_k$ such that

$$E\ v.x_1^{m_1}x_2^{m_2}\ldots x_k^{m_k} \neq 0$$

and similarly to the case $k = 1$ this implies that there exists a $t_*$ close to the origin of $R^k$ such that

$$Ev.\exp(t_*'x) \neq 0.$$

Now let $v_* = v.\exp(t_*'x)$. Since $Ev_* \neq 0$ we have

$$P(E[v_*|x_1,..,x_\ell] = 0) < 1 \text{ for } \ell = 1,..,k.$$

Let for $\ell = 1,..,k$

$$S^*_\ell = \{(t_1,..,t_\ell)' \in R^\ell: Ev_*\exp(t_1x_1+..+t_\ell x_\ell) = 0 \}.$$

We show now that if the lemma holds for $k = \ell$ it holds too for $k = \ell+1$, i.e.

$$P((\hat{t}_1,..,\hat{t}_{\ell+1})' \in S^*_{\ell+1}) = 0.$$

By induction, we then have

$$\text{(A2)}\quad P((\hat{t}_1,..,\hat{t}_k)' \in S^*_k ) = 0,$$

and replacing $\hat{t}$ by $\hat{t}-t_*$ we see that then also

$$P((\hat{t}_1,..,\hat{t}_k)' \in S ) = 0,$$

which is the desired result.

Thus, assume

$$P((\hat{t}_1,..,\hat{t}_\ell)' \in S^*_\ell ) = 0.$$

Then

$$
\begin{aligned}
&P\{(\hat{t}_1,..,\hat{t}_{\ell+1})' \in S^*_{\ell+1}\} \\
&\quad = P\{(\hat{t}_1,..,\hat{t}_{\ell+1})' \in S^*_{\ell+1} \text{ and } (\hat{t}_1,..,\hat{t}_\ell)' \in S^*_\ell\} \\
&\quad + P\{(\hat{t}_1,..,\hat{t}_{\ell+1})' \in S^*_{\ell+1} \text{ and } (\hat{t}_1,..,\hat{t}_\ell)' \notin S^*_\ell\} \\
&\quad = P\{(\hat{t}_1,..,\hat{t}_{\ell+1})' \in S^*_{\ell+1} \text{ and } (\hat{t}_1,..,\hat{t}_\ell)' \notin S^*_\ell\} \\
&\quad = P\{\hat{t}_{\ell+1} \in S^{**}_{\ell+1}(\hat{t}_1,..,\hat{t}_\ell) \text{ and } (\hat{t}_1,..,\hat{t}_\ell)' \notin S^*_\ell\},
\end{aligned}
$$

where

$$S^{**}_{\ell+1}(t_1,..,t_\ell) = \{\ t \in R : E[v.\exp(t_1x_1+..+t_\ell x_\ell)]\exp(t.x_{\ell+1}) = 0\ \}.$$

Since by the argument for the case $k = 1$ it follows that

$$S^{**}_{\ell+1}(t_1,..,t_\ell) \text{ is countable if } (t_1,..,t_\ell) \notin S^*_\ell,$$

and since by the absolute continuity of the distribution of $(\hat{t}_1,..,\hat{t}_k)'$ the conditional distribution of $\hat{t}_{\ell+1}$ relative to $\hat{t}_1,..,\hat{t}_\ell$ is continuous, the desired result (A2) easily follows. Q.E.D.

*Remark:* Observe that the result of the lemma under review implies that the set S has Lebesgue measure zero.

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